Journal of Engineering Physics and Thermophysics, Vol. 83, No. 3, 2010

TRACKING OF THE PRESET PROGRAM OF WEIGHTED TEMPERATURES AND RECONSTRUCTION OF HEAT TRANSFER COEFFICIENTS

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UDC 519.6

An approach to numerical solution of problems of reconstruction of heat transfer coefficients in nonlinear nonstationary initial boundary-value problems based on a preset sum of weighted temperatures has been developed and tested.

Keywords: *inverse problems, optimum control, nonlinear heat conduction equation, reconstruction of heat transfer coefficients, sum of weighted temperatures, regularizer.*

Introduction. Control and diagnostics of thermal processes is an integral past of state-of-the-art technologies in metallurgy, nuclear power engineering, food industry, and other branches of the national economy. The mathematical basis for calculating and modeling the indicated processes consists of methods of the theory of inverse problems and optimum control [1–6].

In the current study, consideration is given to the problem of reconstruction of heat transfer coefficients for processes described by nonstationary nonlinear heat conduction equations. The results can be used for constructing controls in the problem of tracking the preset program of a sum of weighted temperatures or for solving the problems of identifying heat transfer coefficients and boundary heat fluxes from measurements of a sum of weighted temperatures. Specifically, these can be measurements made by an ordinary or differential thermocouple.

It is well known that the examined inverse problems are ill-posed. Monographs [1–5] present iterative methods for solving inverse ill-posed heat conduction problems. We develop an approach based on a suboptimum continuous-discrete method of step-by-step optimization [7, 8]. The proposed algorithm of reconstruction of heat transfer coefficients draws on step-by-step linearization of a nonlinear heat conduction equation, which allows one to solve a series of problems of small dimensionality instead of a problem of large dimensionality. Noisey data are filtered using a regularizer in the form of a function of total variation regularization [6, 9–11].

The presented results of the numerical experiment point to the efficiency of the developed method for numerical reconstruction of heat transfer coefficients under the conditions of accurate and noised data alike.

Problem Statement. We consider a nonlinear heat conduction equation

$$C\,(T\,(x,t))\,\frac{\partial T\,(x,t)}{\partial t}=\frac{\partial}{\partial x}\left(\lambda\,(T\,(x,t))\,\frac{\partial T\,(x,t)}{\partial x}\right) \tag{1}$$

with boundary and initial conditions

$$\left.\lambda\,(T\,(x,t))\,\frac{\partial T\,(x,t)}{\partial x}\right|_{x=0}=u_1\,(t)\,(T_1^{\mathrm{m}}\,(t)-T\,(0,t))\,,\quad t\in[0,t_*]\,, \tag{2}$$

$$\left.\lambda\,(T\,(x,t))\,\frac{\partial T\,(x,t)}{\partial x}\right|_{x=b}=u_2\,(t)\,(T_2^{\mathrm{m}}\,(t)-T\,(b,t))\,,\quad t\in[0,t_*]\,, \tag{3}$$

Institute of Mathematics, National Academy of Sciences of Belarus, 11 Surganov Str., Minsk, 220072, Belarus. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 83, No. 3, pp. 584–592, May–June, 2010. Original article submitted September 16, 2009.

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$$T(x, 0) = T^*(x), \quad x \in [0, b]. \tag{4}$$

The system of equations (1)–(4) describes a distributed dynamic system [5] for which $T = T(x, t)$, $x \in [0, b]$ and $t \in [0, t_*]$, is the function of state of a system, and $u_1(t)$ and $u_2(t)$, $t \in [0, t_*]$, are functions of controls.

It is desired to find control functions (heat transfer coefficients) $u_1(\cdot) = (u_1(t), t \in [0, t_*])$, $u_2(\cdot) = (u_2(t), t \in [0, t_*])$ that satisfy the constraints

$$a_1 \le u_1(t) \le a^1, \quad a_2 \le u_2(t) \le a^2, \quad t \in [0, t_*] \tag{5}$$

and minimize the figure of merit

$$J(u_1(\cdot), u_2(\cdot)) = \max_{t \in [0, t_*]} \left| \sum_{p=1}^{s} d_p T(x_p^*, t) - y(t) \right| \to \inf_{u_1(\cdot), u_2(\cdot)} \tag{6}$$

on the trajectories $T(x, t)$, $x \in [0, b]$ and $t \in [0, t_*]$, (the system of equations (1)–(4)). Here, x_p^*, $p = 1, ..., s$, are specified points from the section $[0, b]$, and a_1, a^1, a_2, and a^2 are specified numbers. The coefficients d_p, $p = 1, ..., s$, characterize weights of the temperature fields at the points x_p^*, $p = 1, ..., s$, and the function $y(t)$, $t \in [0, t_*]$, presets the program of tracking of a sum of weighted temperatures indicated in expressions (6). Specifically, with appropriately selected coefficients we obtain the problem of tracking of the average temperature, and with $s = 1$ and $d_1 = 1$ we obtain the problem of tracking of the temperature at the specified point $x = x_1^*$.

The formulated optimum control problem can be interpreted as the inverse problem of reconstruction or evaluation of heat transfer coefficients $u_1(\cdot)$ and $u_2(\cdot)$ using data for $y(t) = \sum_{p=1}^{s} d_p T(x_p^*, t)$, $t \in [0, t_*]$. In this case constraints (5) have the meaning of a priori information on the sought coefficients. The functions $u_1(t)$, $u_2(t)$, and $T(x, t)$, $t \in [0, t_*]$ and $x \in [0, b]$, obtained from solving the problem can be directly used for solving some other inverse problems. We would like to note two of them. The first pertains to the determination of the heat transfer coefficients $\tilde{u}_i(T)$, $i = 1, 2$, viewed as temperature functions. This problem reduces to solving linear functional equations

$$\tilde{u}_1(T(0, t)) = u_1(t), \quad \tilde{u}_2(T(b, t)) = u_2(t).$$

Provided the functions $T(0, t)$ and $T(b, t)$, $t \in [0, t_*]$, are monotonic we obtain

$$\tilde{u}_i(T) = u_i(g_i(T)), \quad i = 1, 2,$$

where $g_i(T)$, $i = 1, 2$, are functions inverse to functions $T(0, t)$ and $T(b, t)$, $t \in [0, t_*]$, respectively. It should be remarked that in the case of unique solutions the monotonic character of functions $T(0, t)$ and $T(b, t)$, $t \in [0, t_*]$, ensures uniqueness of the solutions for $\tilde{u}_1(T)$ and $\tilde{u}_2(T)$.

The second problem lies in the reconstruction of the heat fluxes $q_1(t)$ and $q_2(t)$ on the boundaries $x = 0$ and $x = b$. Its solution can be represented as $q_1(t) = u_1(t)(T_1^{\mathrm{m}}(t) - T(0, t))$ and $q_2(t) = u_2(t)(T_2^{\mathrm{m}}(t) - T(b, t))$.

Algorithm for Solving the Optimum Control Problem. With the aid of a standard technique an additional variable ξ is introduced and problem (1)–(6) is reformulated in the equivalent form as follows.

Problem 1: find the parameter ξ and the control functions $u_1(t)$ and $u_2(t)$, $t \in [0, t_*]$, satisfying constrains (5) such that on the trajectory $T(x, t)$, $x \in [0, b]$ and $t \in [0, t_*]$, of the system of equations (1)–(4) phase restrictions are fulfilled

$$\left| \sum_{p=1}^{s} d_p T(x_p^*, t) - y(t) \right| \le \xi, \quad t \in [0, t_*],$$

and the parameter ξ takes the value

$$\xi \to \min_{\xi, u_1(\cdot), u_2(\cdot)} . \tag{7}$$

The initial boundary-value problem (Eqs. (1)–(4)) is approximated by a system of ordinary differential equations. For this, the section $[0, b]$ is split into N parts by the points $x_i = ih$, $i = 0, ..., N$, $h = b/N$, the designation $T_i(t) = T(x_i, t)$, $t \in [0, t_*]$, is introduced, and the system of equations (1)–(4) is replaced by the following nonlinear system of ordinary differential equations:

$$\frac{d}{dt} T_0(t) = \frac{u_1(t)\,(T_0(t) - T_1^{\mathrm{m}}(t)) + \lambda(T_0(t))\,(T_1(t) - T_0(t))/h}{hC(T_0(t))}, \quad T_0(0) = T^*(0);$$

$$\frac{d}{dt} T_i(t) = \frac{\lambda(T_i(t))\,(T_{i+1}(t) - T_i(t)) - \lambda(T_{i-1}(t))\,(T_i(t) - T_{i-1}(t))}{h^2 C(T_i(t))}, \quad T_i(0) = T^*(ih), \quad i = 1, ..., N-1; \tag{8}$$

$$\frac{d}{dt} T_N(t) = \frac{u_2(t)\,(T_N(t) - T_2^{\mathrm{m}}(t)) + \lambda(T_N(t))\,(T_N(t) - T_{N-1}(t))/h}{hC(T_N(t))}, \quad T_N(0) = T^*(b).$$

In order for the system of ordinary different equations (8) to satisfactorily approximate the original system of partial differential equations (1)–(4) the number $N > 0$ should be fairly large ($h > 0$ is rather small).

Without loss of generality it is assumed that $x_p^* = i_p h$, $p = 1, ..., s$, and $0 \le i_1 < ... < i_s \le N$. Then, original problem 1 can be approximated by the following problem.

Problem 2: find the parameter ξ and the control functions $u_1(t)$ and $u_2(t)$, $t \in [0, t_*]$, satisfying constrains (5) such that on the trajectory of the system of equations (8) the phase restriction is fulfilled

$$\left| \sum_{p=1}^{s} d_p T_{i_p}(t) - y(t) \right| \le \xi, \quad t \in [0, t_*], \tag{9}$$

and the parameter ξ takes a minimum value.

Problem 2 is the problem of optimum control of nonlinear dynamic system (8) with $(N+1)$-dimensional vector of states $\mathbf{z}(t) = (T_0(t), T_1(t), ..., T_N(t))'$ and controls $u_1(t) \in \mathbb{R}$ and $u_2(t) \in \mathbb{R}$, $t \in [0, t_*]$, with figure of merit (7) in the presence of phase restriction (9) and direct constraints on controls (5). This problem can be written in the form

$$\xi \to \inf_{\xi, u_1(\cdot), u_2(\cdot)},$$

$$\frac{d\mathbf{z}(t)}{dt} = \mathbf{f}\left(\mathbf{z}(t), \mathbf{u}(t), t \,\middle|\, h\right), \quad \mathbf{z}(0) = \mathbf{z}_0, \tag{10}$$

$$\left| \mathbf{e}'\mathbf{z}(t) - y(t) \right| \le \xi, \quad a_1 \le u_1(t) \le a^1, \quad a_2 \le u_2(t) \le a^2, \quad t \in [0, t_*].$$

Here $\mathbf{u}(t) = (u_1(t)$ and $u_2(t))'$, $\mathbf{z}_0 = (T^*(0), T^*(h), ..., T^*(Nh))'$, $\mathbf{e}$ is the vector, whose components e_j, $j = 0, ..., N$, are constructed by the rule $e_{i_p} = d_p$, $p = 1, ..., s$, and the other components are zero; the function $\mathbf{f}(\mathbf{z}, \mathbf{u}, t \,|\, h)$ is readily constructed using the right-hand side of the system of equations (8).

With small $h > 0$, the optimum control problem (10) has some specific features: 1) the vector of the state $\mathbf{z}(t)$ has a larger dimensionality; 2) the function $\mathbf{f}(\mathbf{z}, \mathbf{u}, t \,|\, h)$ is nonlinear as to state and ill-defined for small $h > 0$ since it contains terms of the order of $1/h^2$; 3) the problem has the phase restriction $|\mathbf{e}'\mathbf{z}(t) - y(t)| \le \xi$, $t \in [0, t_*]$, whose index is $k_* = k_*(h) = \min_{p=1,...,s} \min \{i_p, N - i_p\}$.

In the optimum control theory the index (the order) of the phase restriction [12–14] $q(\mathbf{z}(t)) \le 0$, $t \in [0, t_*]$, is said to be a minimum natural number k at which $\frac{\partial}{\partial \mathbf{u}} \frac{d^k q(\mathbf{z}(t))}{dt^k} \ne 0$. Here, derivatives of the function $\mathbf{z}(t)$ with respect to t are calculated on the strength of system (10). The value of the index k characterizes the degree of a "direct" influence of the control function $\mathbf{u}(t)$ on the function $q(\mathbf{z}(t))$ of the phase restriction. At $k = 1$ the control $\mathbf{u}(t)$ can have a direct influence on the first derivative $\frac{dq(\mathbf{z}(t))}{dt}$ of the function of the phase restriction, and at $k = s$ the control $\mathbf{u}(t)$ can have a direct influence only on the derivative $\frac{d^s q(\mathbf{z}(t))}{dt^s}$. The larger the value of the index k, the "weaker" the influence of the control $\mathbf{u}(t)$ on the function of the phase restriction and the more difficult it is to fulfill the phase restrictions. These difficulties are also indicated by the well-known fact [12, 14] that, as a rule, with the index of the phase restriction $k \ge 3$, the optimum control system does not have a solution in the class of piecewise continuous functions. Its solution is a measurable function of complex structure (with the presence of Fuller regimes). It should be noted that for the index $k_*(h)$ of the phase restriction of problem (10) we have $k_*(h) \to \infty$ for $h \to 0$.

The enumerated facts make it impossible to solve problem 2 using computational packages designed for "standard" optimum control problems. We therefore propose a special approximations method taking account of the specifics of problem 2.

Let the time interval $[0, t_*]$ be split into r parts by points $\tau_0 = 0$ and $\tau_j = jh_r$, $j = 1, \ldots, r$, $h_r = t_*/r$. In the proposed algorithm, the solution of the optimum control problem by *nonlinear* system (8) reduces to a consecutive solution of the optimum control problems on the intervals $t \in [\tau_j, \tau_{j+1}]$, $j = 0, \ldots, r-1$. On the interval $[\tau_j, \tau_{j+1}]$, a linear system is obtained from nonlinear system (8) on replacing functions $\lambda(T_i(t))$ and $C(T_i(t))$, $t \in [\tau_j, \tau_{j+1}]$, by the constants $\lambda(T_{ij}^*) = \lambda(T_i(\tau_j))$ and $C(T_{ij}^*) = C(T_i(\tau_j))$. Thus, the problem solved on the jth interval $[\tau_j, \tau_{j+1}]$ is of the form

$$\varepsilon \to \min ;$$

$$\frac{d}{dt} T_0(t) = \frac{u_1(t)\,(T_{0j}^* - T_1^{\mathrm{m}}(t)) + \lambda(T_{0j}^*)\,(T_1(t) - T_0(t))/h}{hC(T_{0j}^*)}, \quad T_0(\tau_j) = T_{0j}^* ;$$

$$\frac{d}{dt} T_i(t) = \frac{\lambda(T_{ij}^*)\,(T_{i+1}(t) - (T_i(t)) - \lambda(T_{i-1,j}^*)\,(T_i(t) - T_{i-1}(t))}{h^2 C(T_{ij}^*)}, \quad T_i(\tau_j) = T_{ij}^*, \quad i = 1, \ldots, N-1 ; \tag{11}$$

$$\frac{d}{dt} T_N(t) = \frac{u_2(t)\,(T_{Nj}^* - T_2^{\mathrm{m}}(t)) + \lambda(T_{Nj}^*)\,(T_N(t) - T_{N-1}(t))/h}{hC(T_{Nj}^*)}, \quad T_N(\tau_j) = T_{Nj}^* ;$$

$$\left| \sum_{p=1}^{s} d_p T_{i_p}(t) - y(t) \right| \le \xi, \quad a_1 \le u_1(t) \le a^1, \quad a_2 \le u_2(t) \le a^2, \quad t \in [\tau_j, \tau_{j+1}] ,$$

where the initial conditions T_{ij}^*, $i = 0, 1, \ldots, N$, are determined by the previous problem solved on the interval $[\tau_{j-1}, \tau_j]$.

We now describe the algorithm of constructing the control $\mathbf{u}(t) = (u_1(t), u_2(t))'$, $t \in [0, t_*]$. Let $j = 0$ and $T_0^*(x) = T^*(x)$, $x \in [0, b]$. For $j \ge 0$, before the beginning of the jth iteration the function $T_j^*(x)$, $x \in [0, b]$, is known and the control $\mathbf{u}(t)$, $t \in [0, \tau_j]$ is constructed. The jth iteration of the algorithm consists of the following steps.

Step 1. We determine the initial state of system (11)

$$T_{ij}^* = T_j^*(x_i), \quad i = 0, 1, \ldots, N . \tag{12}$$

We solve linear optimum control problem (11) on the interval $[\tau_j, \tau_{j+1}]$ with initial state (12). The control $\mathbf{u}(t)$, $t \in [\tau_j, \tau_{j+1}]$ is obtained.

Step 2. With the known value of the control $\mathbf{u}(t)$, $t \in [\tau_j, \tau_{j+1}]$, and the known initial condition $T(x, \tau_j) = T_j^*(x)$, $x \in [0, b]$, we integrate the original nonlinear equation (1) on the interval $[\tau_j, \tau_{j+1}]$. The state of the original system of equations (1)–(4) at the time instant $t = \tau_{j+1} : T(x, \tau_{j+1})$, $x \in [0, b]$, is obtained. We set $T_{j+1}^*(x, \tau_{j+1})$, $x \in [0, b]$.

Step 3. If $\tau_{j+1} = t_*$, the control $\mathbf{u}(t)$, $t \in [0, t_*]$ has been found. Otherwise we set $j := j + 1$.

The main difficulty associated with the implementation of the algorithm lies in solving linear optimum control problems (11) on the intervals $[\tau_j, \tau_{j+1}]$, $j = 0, ..., r - 1$. Account for the specifics of these problems allows one to develop efficient methods of their solution. One of such methods is described in the next section. It should be noted that linearization of the original system of equations (1)–(4) is performed step-by-step on the intervals $[\tau_j, \tau_{j+1}]$. Here, the jth optimum control problem is formed using initial conditions T_{ij}^*, $i = 0, 1, ..., N$, obtained as a result of integrating the original system of equations (1)–(4) along the control found at the previous steps.

Algorithm of the Solution of Problem (11). Evidently, problem (11) can be rewritten in the form

$$P_j(\mathbf{z}^*(j)): \begin{cases} \xi \to \min, \\ \dfrac{d\mathbf{z}(t)}{dt} = \mathbf{A}(\mathbf{z}^*(j))\,\mathbf{z}(t) + \mathbf{B}(\mathbf{z}^*(j))\,\mathbf{u}(t), \quad \mathbf{z}(\tau_j) = \mathbf{z}^*(j), \\ \text{symbol}\,\mathbf{e}'\mathbf{z}(t) - y(t)\,\text{symbol} \le \xi, \quad t \in [\tau_j, \tau_{j+1}], \\ a_1 \le u_1(t) \le a^1, \quad a_2 \le u_2(t) \le a^2, \quad t \in [\tau_j, \tau_{j+1}], \end{cases}$$

where $\mathbf{z}^*(j) = (T_{ij}^*,\ i = 0, 1, ..., N)'$, $\mathbf{z}(t) = (T_0(t), T_1(t), ..., T_N(t))'$, and $\mathbf{u}(t) = (u_1(t), u_2(t))'$; and the matrices $\mathbf{A}(\mathbf{z}^*(j)) \in \mathbb{R}^{(N+1)\times(N+1)}$ and $\mathbf{B}(\mathbf{z}^*(j)) \in \mathbb{R}^{(N+1)\times 2}$ are readily constructed using the right-hand side of system (11), and the vector $\mathbf{e}$ is determined in describing problem (10).

Let us choose the parameter $m \in \mathbb{N}$ and set $\Delta h = (\tau_{j+1} - \tau_j)/m$. The problem $P_j(\mathbf{z}^*(j))$ is considered in the class of piecewise continuous controls

$$u_i(t) = u_{ik} = \text{const}, \quad t \in [\tau_j + k\Delta h, \tau_j + (k+1)\Delta h), \quad k = 0, 1, ..., m - 1, \quad i = 1, 2,$$

and the phase restriction $|\mathbf{e}'\mathbf{z}(t) - y(t)| \le \xi$, $t \in [\tau_j, \tau_{j+1}]$, is replaced by the restriction

$$|\mathbf{e}'\mathbf{z}(\tau_j + k\Delta h) - y(\tau_j + k\Delta h)| \le \xi, \quad k = 0, 1, ..., m - 1.$$

Evidently, the obtained optimum control problem is equivalent to a linear programming problem of the form

$$LP_j(\mathbf{z}^*(j)): \begin{cases} \xi \to \min\limits_{\xi, u_{ik}}, \\ -\xi \le \sum\limits_{i=1}^{2} \sum\limits_{k=0}^{m-1} a_{ik}^{(n)} u_{ik} + \Delta y^{(n)} - y(\tau_j + n\Delta h) \le \xi, \quad n = 0, 1, ..., m, \\ a_i \le u_{ik} \le a^i, \quad k = 0, 1, ..., m - 1, \quad i = 1, 2. \end{cases}$$

Here,

$$\Delta y^{(n)} = \Delta y^{(n)}(\mathbf{z}^*(j), j) = \mathbf{e}'\mu_0(\tau_j + n\Delta h), \quad n = 0, 1, ..., m,$$

$$a_{ik}^{(n)} = a_{ik}^{(n)}(\mathbf{z}^*(j), j) = \mathbf{e}'\boldsymbol{\mu}_{ik}(\tau_j + n\Delta h), \quad n = 0, 1, ..., m, \quad k = 0, 1, ..., m - 1, \quad i = 1, 2,$$

$\mu_0(t) \in \mathbb{R}^{(N+1)}$, $t \in [\tau_j, \tau_{j+1}]$, is the solution of the system

$$\frac{d\boldsymbol{\mu}_0(t)}{dt} = \mathbf{A}(\mathbf{z}^*(j))\,\boldsymbol{\mu}_0(t), \quad \boldsymbol{\mu}_0(\tau_j) = \mathbf{z}^*(j); \tag{13}$$

$\mu_{ik}(t) \in \mathbb{R}^{(N+1)}$, $t \in [\tau_j, \tau_{j+1}]$, is the solution of the system

$$\frac{d\boldsymbol{\mu}_{ik}(t)}{dt} = \mathbf{A}(\mathbf{z}^*(j))\,\boldsymbol{\mu}_{ik}(t) + \mathbf{B}(\mathbf{z}^*(j))\,\boldsymbol{\omega}_{ik}(t), \quad \boldsymbol{\mu}_{ik}(\tau_j) = \mathbf{0}, \tag{14}$$

in which the function $\omega_{ik}(t) = (\omega_{ik}^1(t), \omega_{ik}^2(t))'$ is of the form

$$\omega_{ik}^i(t) = 1, \quad t \in [\tau_j + k\Delta h, \tau_j + (k+1)\Delta h]; \quad \omega_{ik}^i(t) = 0, \quad t \in [\tau_j, \tau_{j+1}] \setminus [\tau_j + k\Delta h, \tau_j + (k+1)\Delta h],$$

$$\omega_{ik}^j(t) = 0, \quad t \in [\tau_j, \tau_{j+1}], \quad j \neq i, \quad i = 1, 2, \quad j = 1, 2.$$

It should be pointed out that dimensions of the problem $LP_j(\mathbf{z}^*(j))$ do not depend on the parameter N and are determined only by the parameter m, which can be chosen small. Integration of the linear systems of differential equations (13) and (14) is carried out fairly accurately and rapidly using standard methods. Specifically, in the numerical experiments we used the MATLAB 7.0 package for solving linear programming problems $LP_j(\mathbf{z}^*(j))$ as well as for integrating systems of differential equations.

Reconstruction of Heat Transfer Coefficients Based on Incomplete and Inaccurate Measurements. We consider the problem of identification of the heat transfer coefficients $u_1^*(t)$ and $u_2^*(t)$, $t \in [0, t_*]$. During the functioning of the system of equations (1)–(4) a measuring device determines a weighted sum of the temperatures $y(t) = \sum_{p=1}^{s} d_p T(x_p^*, t) + w(t)$ with a certain error $w(t)$, $t \in [0, t_*]$. It is desirable to reconstruct the heat transfer coefficients $u_1^*(t)$, $u_2^*(t)$, and $t \in [0, t_*]$, using measurements of $y(t)$, $t \in [0, t_*]$.

In order to solve this problem it is possible to formulate an optimum control problem of the type of problem 2 and to solve it by the method described above. However, in the presence of the noise $w(t)$, $t \in [0, t_*]$, figure of merit (6) should be supplemented by a special term, namely, a regularizer [6]. Various types of regularizers are known in the literature [15–17]. In the current study use is made of the generalization of the function of total variation regularization. With such a type of regularization, the figure of merit of the original problem is supplemented by a term of the form $\gamma \int_0^{t_*} \sum_{i=1}^{k_*} (\beta_1 |u_1^{(i)}(t)| + \beta_2 |u_2^{(i)}(t)|) dt$, where $\gamma > 0$ is the regularization parameter, k_* is the index of the phase restriction, and $\beta_1 \geq 0$ and $\beta_2 \geq 0$ are the specified weights. As a result, figure of merit (7) in problem 2 is replaced by

$$\xi + \gamma \int_0^{t_*} \sum_{i=1}^{k_*} \left(\beta_1 |u_1^{(i)}(t)| + \beta_2 |u_2^{(i)}(t)|\right) dt \to \inf. \tag{15}$$

It should be noted that the quality of noise filtration will definitely be linked to the value of the coefficient γ. Determining an optimum value of γ is a separate problem [16, 17]. In our investigations this value was selected experimentally.

Numerical Experiment. Let us illustrate the functioning of the proposed identification algorithm on model examples in which the function $u_2(t)$, $t \in [0, t_*]$, is assumed to be known. It is desirable to reconstruct the function $u_1^*(t)$, $t \in [0, t_*]$, alone. We therefore set $\beta_1 = 1$ and $\beta_2 = 0$. As the model function (the heat transfer coefficient) $u_1^*(t)$ we choose

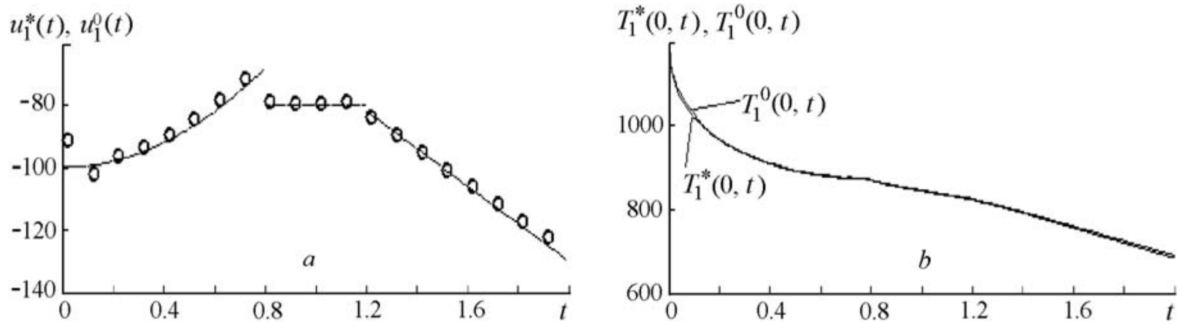

Fig. 1. Results of reconstruction with accurate output data: a) controls $u_1^*(t)$ and $u_1^0(t)$, $t \in [0, 2]$, b) trajectories $T_1^*(0, t)$ and $T_1^0(0, t)$, $t \in [0, 2]$.

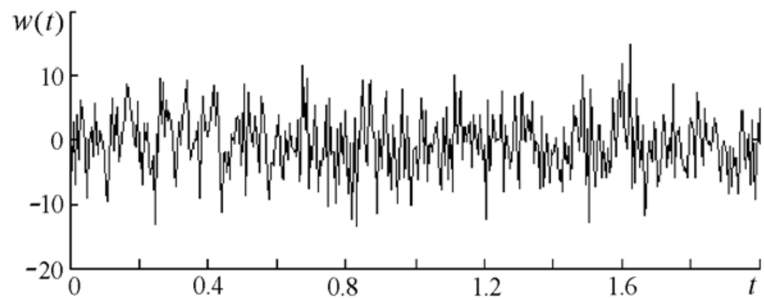

Fig. 2. Realization of discrete white noise according to normal distribution law with an amplitude of 16.

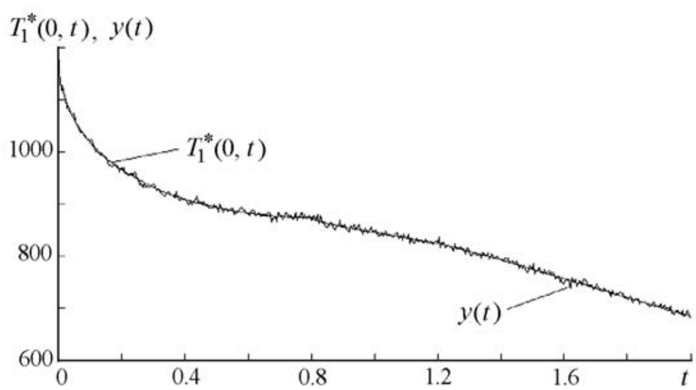

Fig. 3. Accurate and noisey values of the output data for $T^*(0, t)$ and $y(t)$, $t \in [0, 2]$.

$$u_1^*(t) = 50t^2 - 100, \quad t \in [0; 0.8]; \quad u_1^*(t) = -80, \quad t \in [0.8; 1.2]; \quad u_1^*(t) = -60t - 10, \quad t \in [1.2; 2].$$

Furthermore, we assume $s = 1$, $x_1^* = 0$, $b = 0.27$, $t_* = 2$, $N = 27$, $d_0 = 1$, $u_2(t) = -10$, $T(x, 0) = 1200$, $T_1^m(t) = 50$, $T_2^m(t) = 50$; $t \in [0, 2]$; and $\gamma = 4$. Let the functions $\lambda(T)$ and $C(T)$ be of the form

$$\lambda(T) = 0.023T + 34.67, \quad T \in [30^{\circ}C, 700^{\circ}C], \quad \lambda(T) = -0.045T + 81.82, \quad T \in [700^{\circ}C, 1200^{\circ}C],$$

$$C(T) = (4/3)T + 800, \quad T \in [30^{\circ}C, 700^{\circ}C], \quad C(T) = -(8/7)T + 2500, \quad T \in [700^{\circ}C, 1200^{\circ}C].$$

Let $u_1^0(t)$, $t \in [0, t_*]$, denote the reconstructed control function and let $T^*(x, t)$, $x \in [0, b]$ and $t \in [0, t_*]$, and $T^0(x, t)$, $x \in [0, b]$ and $t \in [0, t_*]$, denote trajectories of the system of equations (1)–(4) generated by the controls $u_1^*(t)$, $t \in [0, t_*]$, and $u_1^0(t)$, $t \in [0, t_*]$, respectively.

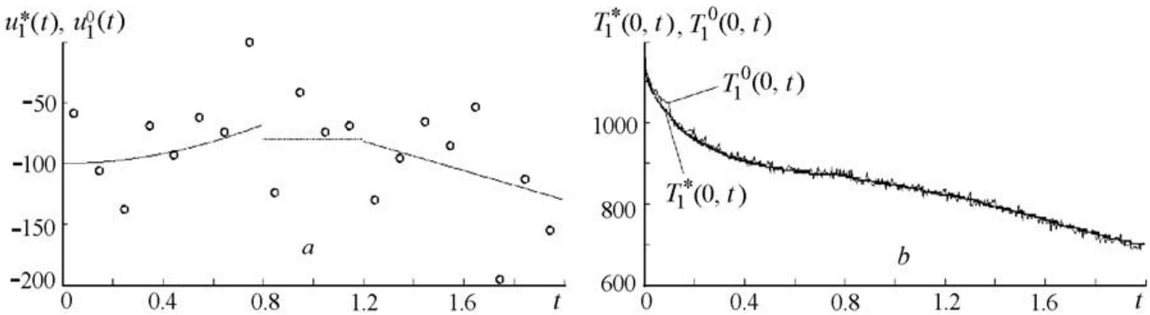

Fig. 4. Results of reconstruction with noisey output data (the noise amplitude is 16) without regularization: a) controls $u_1^*(t)$ and $u_1^0(t)$, $t \in [0, 2]$, b) trajectories $T_1^*(0, t)$ and $T_1^0(0, t)$, $t \in [0, 2]$.

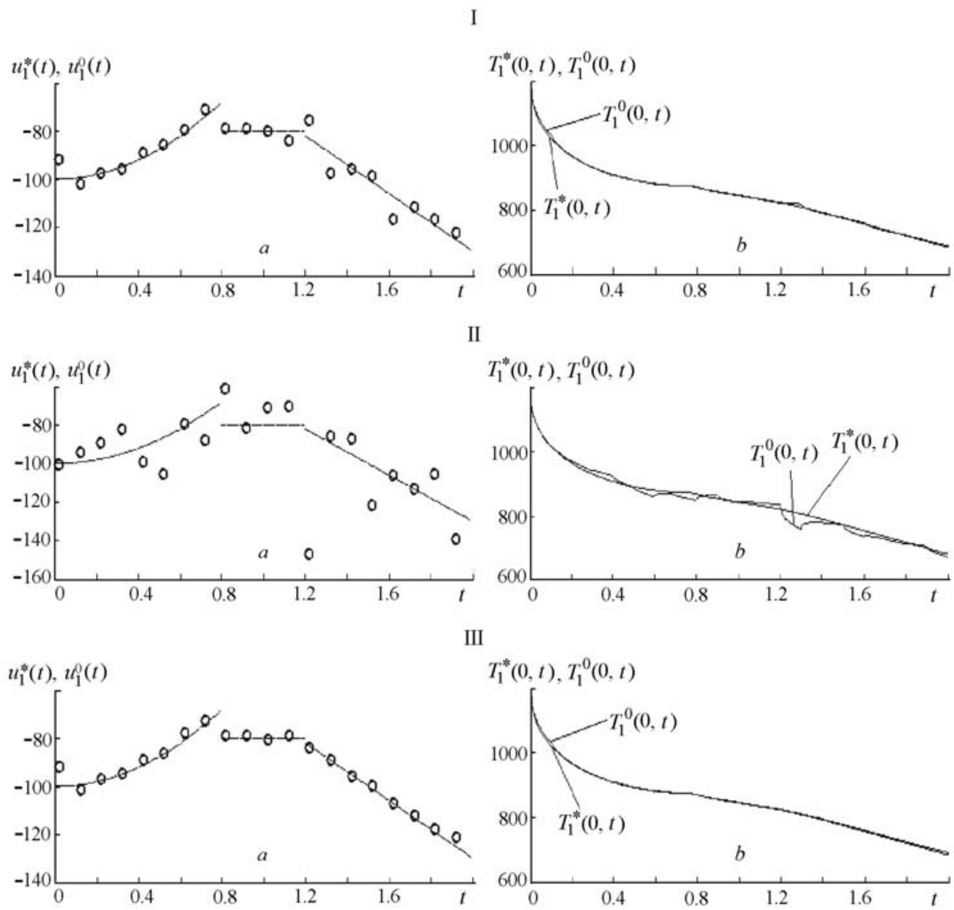

Fig. 5. Results of reconstruction with noisey output data with account for the regularizer at $\gamma = 4$: a) $u_1^*(t)$ and $u_1^0(t)$, $t \in [0, 2]$, b) trajectories $T_1^*(0, t)$ and $T_1^0(0, t)$, $t \in [0, 2]$; I) noise amplitude of 16; II) noise amplitude of 95; III) noise amplitude of 2.5.

Measurement data are determined by the equation $y(t) = T^*(0, t) + w(t)$, $t \in [0, t_*]$, where $w(t)$, $t \in [0, t_*]$, is the function modeling errors in the measurements. For modeling $w(t)$, $t \in [0, t_*]$, use was made of realizations of Gaussian noise of various powers.

It is pointed out that the function $u_1^*(t)$, $t \in [0, 2]$, is discontinuous at the points $t = 0.8$ and $t = 1.2$. On the other hand, the index of the phase restriction is unity, which corresponds to a minimum degree of irregularity of the inverse problem.

We now present results of the numerical calculations and their brief discussion.

1. Let us check the functioning of the algorithm at $w(t) = 0$, $t \in [0, t_*]$, i.e., in the case of accurate measurements. Results of the functioning of the algorithm for this case with $r = 20$ are given in Fig. 1. It should be recalled that r is the number of the solved linear problems of optimum control (11). Here and hereafter figures *a* show the model control $u_1^*(t)$, $t \in [0, 2]$ (a solid curve), and the reconstructed control $u_1^0(t)$, $t \in [0, 2]$ (dots), and figures *b* show the corresponding trajectories $T^*(0, t)$ and $T^0(0, t)$, $t \in [0, 2]$, of the system of equations (1)–(4).

2. Consider the case where the noise $w(t)$, $t \in [0, t_*]$, has an amplitude of 16 (see Fig. 2). The corresponding functions $T^*(0, t)$, $y(t)$ and $t \in [0, 2]$, are given in Fig. 3.

Attempts were made to reconstruct the control $u_1(t)$, $t \in [0, 2]$, under the conditions of noisey output data for $y(t)$, $t \in [0, 2]$, with no regularizer ($\gamma = 0$). Results are presented in Fig. 4.

The result of the functioning of the algorithm with account for the regularizing term in the figure of merit with $\gamma = 4$ is given in Fig. 5,I.

3. We next present calculations in the presence of a "high" level of the noise. Let the noise $w(t)$, $t \in [0, t_*]$, have an amplitude of 95. Figure 5,II shows the result of reconstruction of the function $u_1^*(t)$, $t \in [0, t_*]$, using a regularizer at $\gamma = 4$.

4. The results of reconstruction under the assumption of a "low" level of the noise $w(t)$, $t \in [0, t_*]$ (the amplitude is 2.5), are given in Fig. 5,III.

Conclusions. The calculated results for the index of the phase restriction k_* equal to unity and the numerical experiment for $k_* > 1$ indicate that the proposed method of reconstructing heat transfer coefficients is efficient for both accurate and noisey data.

NOTATION

a_1, a_2, lower bounds of variation of the control functions $u_1(t)$ and $u_2(t)$, respectively, $W/(m^2 \cdot {}^oC)$; a^1, a^2, upper bounds of variation of the control functions $u_1(t)$ and $u_2(t)$, respectively, $W/(m^2 \cdot {}^oC)$; b, right boundary of the section $[0, b]$ of variation of the spatial coordinate, m; $C(T)$, specific heat, $W \cdot h/(m^2 \cdot {}^oC)$; d_p, $p = 1, ..., s$, weight coefficients; k_*, index of the phase restriction; t, running time instant, h; t_*, final time instant, h; $T(x, t)$, temperature field, ^oC; $T^*(x)$, initial temperature, ^oC; $T_1^m(t)$, $T_2^m(t)$, ambient temperature at the left and right ends of the section [0, b], ^oC; $u_1(t)$, $u_2(t)$, heat transfer coefficients (controlling agencies) at the left and right ends of the section $[0, b]$, $W/(m^2 \cdot {}^oC)$; x, spatial coordinate, m; x_p^*, $p = 1, ..., s$, points of the temperature measurement, m; $\lambda(T)$, thermal conductivity, $W/(m^2 \cdot {}^oC)$. Subscripts and superscripts: 1, left boundary; 2, right boundary; s, number of points of the temperature measurement; N, number of points of spliting of the spatial section $[0, b]$; r, number of points of spliting of the time interval $[0, t_*]$; m, medium; $'$, transpose operation.

REFERENCES

1. O. M. Alifanov, E. A. Artyukhin, and S. V. Rumyantsev, *Extremal Methods of Solving Ill-Posed Problems* [in Russian], Nauka, Moscow (1988).
2. O. M. Alifanov, *Identification of Heat-Transfer Processes in Aircraft* [in Russian], Mashinostroenie, Moscow (1979).
3. M. N. Özisik and H. R. B. Orland, *Inverse Heat Transfer: Fundamentals and Applications*, Taylor and Francis, New York (2000).
4. W. H. Ray, *Advanced Process Control Methods* [Russian translation], Mir, Moscow (1993).
5. V. T. Borukhov, I. V. Gaishun, and V. I. Timoshpol'skii, *Structural Properties of Dynamic Systems and Inverse Problems of Mathematical Physics* [in Russian], Belaruskaya Navuka, Minsk (2009).
6. A. N. Tikhonov and V. Ya. Arsenin, *Methods of Solving Ill-Poised Problems* [in Russian], Nauka, Moscow (1979).
7. F. Troltzsch and K. Eppler, Fast optimization methods in the selective cooling of steel, in: M. Grötschel, S. O. Krumke, and J. Rambau (Eds.), *Online Optimization of Large-Scale Systems*, Springer-Verlag, Berlin (2001), pp. 185–204.

8. K. Eppler and F. Troltzsch, Fast solution of optimal control problems in the selective cooling of steel, *ZAMM*, **81**, No. 7, 447–456 (2001).
9. L. I. Rudin, S. Osher, and E. Fatemi, Nonlinear total variation based on noise removal algorithms, *Physica D*, **60**, 259–268 (1992).
10. P. S. Vassilevski and J. G. Wade, A comparison of multilevel methods for total variation regularization, *Electron. Trans. Numer. Anal.*, **6**, 255–270 (1997).
11. L. He, M. Burger, and S. Osher, Iterative total variation regularization with non-quadratic fidelity, *J. Math. Imag. Vision*, **26**, No. 1–2, 167–184 (2006).
12. K. Malanowski and H. Maurer, Sensitivity analysis of optimal control problems subject to higher order state constraints, *Annals of Operations Research* (Optimization with Data Perturbations II), **101**, 43–73 (2001).
13. A. P. Afanas'ev, V. V. Dikusar, A. A. Milyutin, and S. A. Chukanov, *Necessary Condition in Optimum Control* [in Russian], Nauka, Moscow (1990).
14. V. V. Dikusar and A. A. Milyutin, *Qualitative and Numerical Methods in the Maximum Principle* [in Russian], Nauka, Moscow (1989).
15. A. N. Tikhonov, On solution of ill-posed problems and the regularization method, *Dokl. Akad. Nauk SSSR*, **151**, No. 3, 501–504 (1963).
16. D. Krawczyk and M. Rudnicki, Regularization parameter selection in discrete ill-posed problems — the use of the u-curve, *Int. J. Appl. Math. Comput. Sci.*, **17**, No. 2, 157–164 (2007).
17. M. E. Kilmer and D. P. O'Leary, Choosing regularization parameters in iterative methods for ill-posed problems, *Siam J. Matrix Anal. Appl.*, **22**, No. 4, 1204–1221 (2001).